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THE GRAPH IN HIGH-SCHOOL MATHEMATICS

N. J. LENNES
Wendell Phillips High School, Chicago

This paper is to be regarded as "opening the discussion" of a paper by Professor Moore in this number of the *School Review.*[1] I take my text from the first paragraph of that paper:

(1) Pure mathematics is, as it were, a language for the convenient expression and investigation of relations the most diverse in ordinary life and in nature. (2) The principles of the language are not arbitrary, but are imposed by the *phenomena demanding convenient expression.*[2]

The present paper is confined entirely to secondary mathematics.

In order that mathematics shall be, in the hands of the student, "a language for the convenient expression and investigation, etc.," it is necessary that mathematics shall constantly be used as a language for these purposes, or, more properly, it must be largely *developed* as such language. To speak in particular of algebra: It is certainly true that algebra may be so taught—indeed, in very many instances is so taught—that a formula has little more meaning to the average student than have the mystic symbols of the Masonic Order to the uninitiated. Indeed, there are teachers who stoutly maintain that the best way to teach elementary algebra is to teach the formal operations and formulæ as such, without any serious attempt to give exterior interpretations of them. There are others, however—and the writer is one of these—who believe that the great problem in teaching algebra is to connect its operations and formulæ with life—with the life of the student; to give them meaning in terms of those basic concepts which have already been woven into the learner's mind—concepts which are permanent parts of his intellectual equipment. Two such fundamental concepts are the concepts of space and of number. Professor Moore, in the paper under discussion, is emphasizing the importance of the space concept.

[1]It is taken for granted that those who read this paper have read that of Professor Moore. Hence that paper is not here characterized in full.

[2]The italics and the numbering are mine.

I. NUMBER IN ALGEBRA

I wish first to say a word about the number concept in elementary algebra. One who is not directly engaged in secondary instruction in mathematics may well assume, as a matter of course, that algebra is connected most thoroughly with numbers; its subject-matter is numbers, and no connection is needed. In practice this is not so. Many pupils (probably the majority of them) have no clear idea of the meaning of the equation $ax=b$, and the solution $x=\infty\frac{b}{a}$. I have repeatedly given pupils in the third year of the high school the problem of multiplying 2^7 by 2^9. From 90 to 100 per cent. of the members of these classes get the result 4^{16}. If then they are asked to multiply a^7 by a^9, the answer is unanimously a^{16}. I once remarked on this startling phenomenon to a college professor, who assured me that I must be mistaken as to the facts; while a teacher of mathematics in a prominent high school seriously offered the explanation that pupils have not yet learned to generalize! Probably the most urgent need of secondary mathematics at present is the definite and conscious building of algebra as generalized arithmetic.

A small matter like consistently using the word "number" instead of the word "quantity" would help. The laws of operation in algebra should almost invariably be developed with numbers in the Arabic (Hindu) notation, and then set down in the literal notation to indicate that the *operations* are the same no matter what *numbers* are used.

II. THE GRAPH IN ALGEBRA

By "graph" I mean here any figure usually constructed on ruled (square) paper, which represents phenomena subject to mathematical treatment, and which yields certain conclusions by means of the evident (assumed) properties of the figure. All the properties of the ruled paper are assumed without formal discussion. Intuition and experimentation are given full sway.

Probably one of the most common graphs used in algebra is a rectangle to exhibit the product of two polynomials. To teach the multiplication of polynomials I have found it convenient to proceed as follows: The product $(2+4)(3+5)=6+10+12+20=48$ *because* $6\times8=48$. Illustrate this by a figure. Work a number of problems

of this kind, keeping the problems thus worked out in a collection (on the board) for inspection. By a series of questions the following description of the process can then be elicited: "To obtain the product of two polynomials, multiply every term of one polynomial by every term in the other, and add the products." Then this rule or description of the process is abbreviated into,

$$(a+b+c+\ldots)(e+f+g+\ldots)=ae+af+\ldots+be+bf+\ldots\ldots\ldots.$$

In this way the formula comes to represent well-understood facts in arithmetic and in geometry; it is a live formula, not a mystic symbol. With proper guidance and encouragement pupils will now construct figures for the products, $(a+b)^2$, $(a-b)^2$, and $(a+b)(a-b)$.

The following may be set down as a general formula for teaching algebraic processes: (1) Perform the operation using plain numbers (Arabic notation). (2) Whenever convenient, construct figures representing the same operation. (3) Write out a description of the process in ordinary English. (4) Abbreviate this description into a *formula.*

But algebraic processes should not be made the chief, nor even the sole, end in teaching elementary algebra. However beautiful and interesting these may appear to the accomplished mathematician, the young pupil finds them of little interest in themselves. The processes acquire interest and importance only when by means of them the pupil is enabled to solve problems which without them would be quite beyond him. What is needed is a set of interesting problems. By using the graph and the intuition freely, we obtain a body of geometric theorems which enable us to apply the most elementary algebra to the solution of highly interesting problems. The theorem of Pythagoras may be proved graphically (the style of proof used applies whenever the two sides of the triangle are commensurable). The altitude, and hence the area, of a triangle can then be obtained when the sides are given. The general case when the lengths of the sides are given in terms of letters is a fine problem for a first-year boy in a high school.

A certain type of good, dutiful student will work faithfully manipulating complicated expressions for the sake of obtaining the answer in the back of the book. A different type of student, usually much abler, will attack a problem like the general solution of the triangle

with all his might, because in solving this problem he obtains the area of *every triangle* at one fell swoop, provided its sides are given. The *problem* is the real thing under consideration, not the process—a problem whose nature is perfectly plain and which is full of interest. The algebraic operations needed for its solution become exceedingly interesting *because* they do things which the pupil wants very much to do, and which he cannot do without them.

The altitude, volume, etc., of a regular tetrahedron whose sides are given can now be obtained (the formula for the volume is, of course, assumed), and furnish the best subject-matter I know in connection with which to study radicals. First use a tetrahedron whose edge is some definite number, say 6; then solve one whose edge is 8, etc.; and then solve the problem in the general case where the length of the edge is a. This gives rise to nearly all the processes in radicals which are taught in the high schools.

In connection with simultaneous equations the graph is of paramount importance. The way to do is not to graph the equation, but to graph the condition which gives rise to the equation. As Professor Moore remarks: "With the cross-section paper in commonplace use, the boy of nine will readily understand and create diagrams of train motions, and will enjoy making the limited express overtake the slow freight at a certain time and place." The boy of thirteen goes farther. He writes a statement which gives the relation between the time since the train started (or some other fixed time) and the distance the train has traveled. He understands that this statement applies to any length of time—that his statement gives a relation between two *variables*, and that this relation is satisfied by an infinite number of pairs of numbers. He then writes an abbreviation of this statement, thus obtaining an *equation* which states the same relation. Thus he has two complete descriptions of the motions of the train, one being the line and the other the corresponding equation. Two lines, representing the motion of two trains, give him the means, as it does the boy of nine, to determine the time and place where the fast train overtakes the slow one; but again the boy of thirteen goes farther. He learns how to manipulate the two descriptions (equations) of the lines so as to obtain from them another solution of his problem. A boy so trained will not speak of *the* value of x and *the*

value of y in an equation $3x+4y=12$. He will not talk nonsense about such things, because to him they are simple common-sense.

In this manner I have had boys and girls in the first year of high school graph the relation between the readings of the different thermometers and obtain the corresponding equations. They understand perfectly well that the graphs and the equations serve to give the reading of one thermometer when that of the other is given. In this, as in most cases, the making of the graph is the simplest part of the work, and should precede the making of the equation.

The above indicates in outline the graphical work that I have undertaken to do with first-year classes.[1]

In a more advanced course in algebra taken in the third or fourth year in the high school the graph is again taken up. It is now possible (if one wishes to do so) to prove that every equation of the type $ax+by+c=0$ is the equation of a straight line (of late I have omitted this proof). The question of the simultaneity of linear equations containing two variables can now be discussed with ease. By analogy we may then also discuss as to simultaneity of the case of three linear equations.

When we come to deal with simultaneous equations of degree higher than the first, the graph is a veritable gold mine. I speak in detail only of the case

$$\begin{cases} ax+by+c=0 \\ x^2+y^2=r^2 \end{cases} \tag{1}$$

It is very easy for the pupil to convince himself that the equation $x^2+y^2=25$ represents a circle of radius 5 having its center at the origin. The correspondence of the solutions of a pair of simultaneous equations and the intersections of the corresponding graphs are by this time familiar, and we can predict from the graph that, if the algebra tells the whole story, then there must be two distinct pairs of numbers which satisfy both of the equations (1). We are further able to predict the existence of coincident roots; that is, there are lines tangent to the circle, and if we obtain the equation of one such line, and solve it simultaneously with $x^2+y^2=25$, we shall get only

[1]In the mathematical library at the University of Chicago there are notebooks containing the graphical work of first-year classes. These notebooks are faithful copies which pupils have made of their own books.

one solution. It is also easy to see that there is no pair of number which will satisfy the equations $x+y=10$ and $x^2+y^2=25$. Hence the question of imaginary numbers arises in a natural way. (I have thought best not to go into any detailed discussion of the imaginary number in a high-school course. We have simply interpreted the imaginary values obtained when solving the equations as an indication that no real [ordinary] numbers exist which satisfy the conditions imposed.)

The converse problem is of the highest interest, and is of great usefulness in clarifying the meaning of algebraic expressions. Equations (1) are solved simultaneously. The problem then is to substitute for a, b, c, and r such numbers that (a) the line and the circle shall meet at two distinct points; (b) the line shall be a tangent to the circle; (c) the line shall have no point in common with the circle. Predictions are made for certain numbers, and then the lines and the circles are constructed. The straight line is then treated in a similar manner with tbe parabola, the ellipse, and the hyperbola. The solution of a pair of equations like

$$ax+by+c=0$$

$$\frac{x^2}{H^2}+\frac{y^2}{K^2}=I$$

affords excellent practice in solving quadratics, and the pupil sees good reason why he should obtain the solution. The circle and the ellipse treated simultaneously afford a good example where there are four pairs of numbers satisfying the equations.

I summarize the advantages of graphs in algebra as follows:

1. The subject is more interesting throughout.

2. The subject-matter is more clearly understood.

3. The algebra is connected with many other phases of intellectual activity. It becomes "a language, etc." Hence it has an importance to pupils other than as a requirement for graduation.

4. The existence of other branches of mathematics is pointed out, and these are made partly intelligible. Hence it furnishes an incentive to the further study of mathematics.

5. The pupil emerges from the study of algebra with a higher conception of "understanding a subject" than he otherwise would.

Understanding is not so likely in his case to mean "remembering" (as it so often does). It means "insight."

6. No time is lost. It does not require a whit more time to teach simultaneous equations by means of graphs than without them.

The greater interest and the comparative ease with which the subject is grasped more than compensate in point of time for the few recitations required to master the graph.

III. THE GRAPH IN GEOMETRY

In the language of Professor Moore: "Indeed, for the purpose of elementary education our *current deductive geometry is of the nature of a fetish* to be abandoned in favor of a geometry built on a system of axioms rich enough to validate at once the constructional method of the square ruled paper."

The situation seems to be something like this: Pupils who begin the study of deductive geometry have had no experience with logical deduction as such. They have been in the habit of convincing themselves, by means of *all* the resources at their command, that certain statements are or are not valid. To them it is bewildering in the extreme to be asked to "prove" a certain statement by means of certain other statements, and to be careful not to let intuition enter into the proof in the slightest degree. They are required, in the short space of a month or two, to become quite adept, not only in a new subject, but in a new kind of activity—a kind of activity which in itself is very difficult, and which is entirely strange to the pupil. The wonder to me is that they do as well as they do. One does not know whether to admire or pity the gentle docility of a boy of fourteen who struggles gamely to prove that all right angles are equal. He is as sure of the validity of the proposition as he is sure that he is alive. He no more grasps the meaning of "logical deduction from certain assumed axioms" than he did when he was ten years younger.

As a matter of fact, we all fail to carry out the avowed Euclidian program in our practice. We do better than we pretend. Our proofs, especially in the first part of the course, are by no means strictly deductive. Indeed, they are much farther from being so than is currently supposed.

It is proposed, as I understand it, that, instead of carrying out the

Euclidian program from the beginning, we shall consciously adopt a quite different course. My own practice has been as follows:

a) Write down a list of propositions about space (in terms of points and lines in a plane). This list includes all the statements we can find about whose validity we are all agreed there can be no doubt. Every statement about which there is a doubt is ruled out. The certainty as to their validity must arise from what we all see *has to be so*, and not from some extraneous authority. These propositions should be carefully worded and numbered. (We refer to this list later on as list (A).)

b) In this process we soon come to propositions of which we are not certain. (Most of the propositions which we discuss are found in our text as theorems, corollaries or exercises.) Then we may satisfy ourselves of their validity by careful construction. Since we have the square paper, all its obvious properties being included in list (A), we can now begin to mention that the properties in question are *exactly* true, if the propositions of (A) are exactly true, much more exactly true than we can ever verify by actual construction. "It must be so," says the pupil. That is his first "proof;" but little does he recognize it as that suggested or written out in full in the text. In this manner we proceed learning geometry. Intuition, construction, argumentation, all have their natural sway. But argument (proof) becomes increasingly important as we go on. A list is kept of these propositions which are established by construction. (We refer to this list here as list (B).) It is surprising how soon list (B) is completed. We get so that we can *see* whether propositions are true or not without constructions, i. e., we are *proving* them. List (A), however, continues to receive additions throughout the year.

Great freedom is given as to the means of proof. If an algebraic proof is more convenient than a so-called geometric proof, then use algebra. If the use of algebra involves new propositions, we only need to make provisional cr permanent additions to list (A). *Let the pupil have his way* about the kind of proof he is to use. The great thing is to find a pupil who has a *way* at all. The less hopeful ones are those who are always waiting for the teacher's way. Never stop one pupil who is making a proof because another pupil insists that he has a shorter and more elegant one. If possible, get several

proofs. The pupil who has worked out a difficult one will understand a simpler one easily enough.

When the number of pages allotted for the year's work have been covered in this manner, the pupil has *lived* himself into somewhat of an understanding of what is meant by logical deduction. If thought desirable, the question can now be raised as to whether some of the propositions of the lists (A) and (B) cannot be logically deduced from the remaining ones.[1] If this is not done, there will be considerable time left for working out special problems, every one of which requires deduction from theorems already proved.

I sum up the advantages of such a course as follows:

1. The pupil is not required to spend a good share of his time trying to do what he cannot possibly do.

2. He feels vastly more cheerful, which is a great desideratum both from the point of view of good health and from the point of view of possible progress. He is not quite so ready to shake the dust of mathematics off his feet.

3. He gets acquainted with a larger body of mathematics, and sees more of its relations to other subjects, because he has more time.

4. He is aware of the existence, among other problems, of a large problem in logical argumentation, viz., the reduction of the lists (A) and (B) to the smallest possible proportions. He has not finished the subject.

IV. THE NEGATIVE NUMBER IN ALGEBRA

"The principles of the language are not arbitrary, but are imposed by the phenomena demanding convenient expression." (See p. 317.) Under this, the second part of our text, I wish merely to say a word about the negative number. In trying to define algebra so as to distinguish it from elementary arithmetic, it is often said that the presence of the negative number is the distinguishing characteristic of algebra. This may be true formally, but in reality the negative number is present in arithmetic. Instead of speaking of positive and negative latitude, we speak of latitude north and south of the equator. In finding the difference between two latitudes, one north and the

[1] In the early part of the year this would have been a useless question, since the pupils did not know how to deduce. And they could no more learn by being told than they could learn to swim or ride the bicycle that way.

other south, we do exactly the same thing that we do when we find the difference between a positive and a negative number. It so happens that in dealing with many different phonomena in our environment it is convenient to start at a certain point, not an end point, and measure, so to speak, in two directions. It is for the "convenient expression and investigation" of such phenomena that the negative number is introduced in elementary algebra. The laws of operation with negative numbers are best obtained, for the purposes of instruction, solely by induction from concrete examples to which the negative number is applied. When we once admit that the principles of mathematics are imposed by the "phenomena demanding convenient expression," this mode of treating the negative number is obvious. I give just one example to illustrate the mode of procedure.

$$500 \text{ gain} + 300 \text{ loss} = 200 \text{ gain}$$

When *translated* into the language of algebra (under the convention that a positive number shall represent gain and a negative number shall represent loss), we have

$$+500 + (-300) = +200$$

CONCLUSION

As in geometry, the widespread belief that our current texts are rigorously and infallibly logical throughout has, in my judgment, done much to retard the progress of teaching; so in algebra, the feeling that the laws of operation derive their sanction from on high rather than from man himself, and through him from his environment, has done much to keep us from taking a common-sense view of the situation. As in many other cases, so here, the simplest point of view is also the most profound.

I have commented on Professor Moore's paper so far as I have had experience with the things he proposes. Everything mentioned in this paper has been tried repeatedly in classes. I have tried a good many other tricks with graphs, etc., which at the time I thought were fine; but as the deepest-laid plots often fail, so our finest appearing plans will at times go wrong. This has repeatedly been my experience. I have been able to make progress but slowly. In the course of another year I may be able to fit into some course the

parabola $y=x^2$ for the purposes of construction somewhat as suggested by Professor Moore. It is always my feeling that we are most truly reformers—we are most truly followers of the champions of reform—when we become imbued with the feeling that things may and will be made better than they are at present. It even sometimes happens that, to be the followers in the highest sense of those who are pointing the way to better things, we are compelled to do certain specific things which are diametrically opposed to those they advocate. Every individual teacher must become a center of reform, in the main working out his own salvation. Suggestions and stimulus from others are of course indispensable, but the suggestions can seldom be worked out as they were intended by him who gave them.

I wish to say, finally, that as a result of six years' experience in teaching elementary mathematics in a high school, I am firmly convinced that graphical work is of very great importance in creating interest, and in promoting a clearer and, to the student, a more satisfying insight into subjects which but too often are mysterious riddles.